J. Fluid Mech. (1987), vol. 174, pp. 381–397
Printed in Great Britain

On the ability of drops to stick to surfaces of solids. Part 3. The influences of the motion of the surrounding fluid on dislodging drops

By E. B. DUSSAN V.

Schlumberger-Doll Research, Old Quarry Road, Ridgefield, CT 06877, USA

(Received 12 July 1985 and in revised form 20 May 1986)

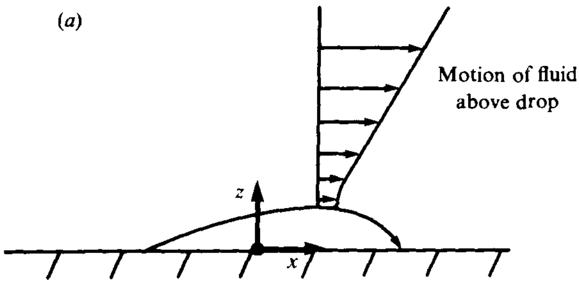
The ability of a drop to stick to a solid surface is investigated when the surrounding fluid is in motion. The specific problem analysed consists of a small drop on a planar surface immersed in a second immiscible fluid which is flowing parallel to the solid surface at a constant rate of strain. An expression is obtained, in terms of experimentally measurable quantities, for the value of the rate of strain beyond which the drop cannot maintain contact with a fixed position on the solid. The most limiting restrictions assumed in the analysis are that both the advancing contact angle and the contact angle hysteresis must be small.

1. Introduction

Drops sticking to the surfaces of solids are encountered in numerous situations. Their presence can be desirable or detrimental depending upon the particular circumstance. In either case, it would be of value to know the various critical conditions beyond which the drops cease to stick. Part 1 (Dussan V. & Chow 1983), and Part 2 (Dussan V. 1985) of this series have focused on the influence of gravity in dislodging the drops from non-horizontal surfaces. The objective of this investigation is to evaluate the ability of the motion of the surrounding fluid to remove the drops by sweeping them across the solid surface.

The specific problem analysed consists of a drop on a surface surrounded by an immiscible fluid undergoing a motion given by $\boldsymbol{u} = (\mathrm{d}U/\mathrm{d}Z)\, Z\boldsymbol{i}$, excluding the immediate vicinity of the drop, where $\mathrm{d}U/\mathrm{d}Z$ denotes a constant (see figure 1*a*). The location of the drop remains stationary for small values of $\mathrm{d}U/\mathrm{d}Z$; however, upon exceeding a *critical value*, the drop moves continuously across the surface, no longer able to remain attached to the solid at one particular position. The aim of this study is to determine the critical value of $\mathrm{d}U/\mathrm{d}Z$ for a given system of materials in terms of experimentally measurable quantities. It is assumed that this problem captures the essence of the more general problem consisting of a rather small drop on a solid surface well embedded within the linear portion of the velocity field of the surrounding fluid. It is of practical interest to know the speed at which the surrounding fluid must flow so that small drops are dislodged and swept away.

The physical characteristic responsible for the drops appearing to stick to the solid surface is the hysteresis in the contact angle. This refers to the experimental observation that contact lines do not move when the value of the contact angle, Θ, lies within the interval (Θ_R, Θ_A). In this study the contact angle will be evaluated from within the drop. Hence, the drop would advance forward displacing the surrounding fluid from the solid surface at those positions along the contact line for



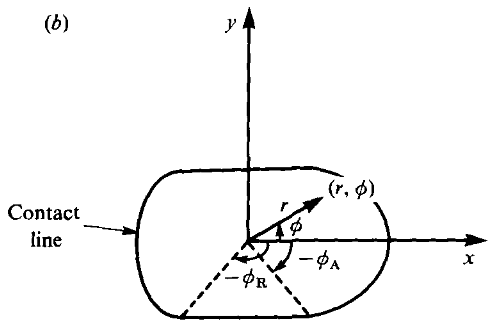


FIGURE 1. A side view of the drop is given in (*a*). The y-axis points into the page. A plan view of the surface of the solid is given in (*b*).

which Θ is greater than Θ_A, often referred to as the *advancing contact angle*; and conversely, the drop would recede, being removed from the solid surface and replaced by the surrounding fluid, at those positions along the contact line for which Θ is less than Θ_R, the *receding contact angle*.

As in the two previous studies, it is assumed that the contact line has two straight-line segments along the sides of the drop when it is in its critical configuration (see figure 1*b*). It is also assumed that $\Theta = \Theta_A$ for $-\phi_A < \phi < \phi_A$, and $\Theta = \Theta_R$ for $\phi_R < \phi < 2\pi - \phi_R$. [Refer to Dussan V. & Chow (1983) for a comprehensive justification of these assumptions.] Hence, the boundary-value problem to be solved contains boundary conditions of the mixed kind. That is to say, along the straight-line segments of the contact line, the local values of the contact angles are not known *a priori*; while, along the remainder of the contact line, the variation in the value of the contact angle is given, with the location of the contact line being a part of the solution. The problem is further complicated by the fact that the locations of the ends of the straight-line segments, i.e. ϕ_A and ϕ_R, are also part of the solution.

There are some significant differences between the current problem and that presented in Parts 1 and 2. In the previous studies, the fluids are static when the drop is in its critical configuration; however, in the present investigation the fluids are in motion. Hence, an analysis of the velocity field is necessary in order to determine the critical value of dU/dZ. It also follows that the dependent variables scale differently.

Another important difference between the two sets of problems lies in the information that can be deduced by performing a macroscopic force balance on the entire drop. In the case of a drop on a non-horizontal surface, the force balance yields

$$(\rho_d - \rho_s) g V \sin\gamma_c = \omega\sigma(\cos\Theta_R - \cos\Theta_A),$$

where ρ_d, ρ_s, g, V, ω, σ, and γ_c denote, respectively, the densities of the drop and surrounding fluids, the gravitational constant, the volume of the drop, the width of the drop, the surface tension, and the angle of inclination of the solid with respect to the horizontal. Hence, the value of γ_c can be thought of as playing a role similar to that of dU/dZ in the current problem. Strictly speaking, this expression is not predictive because the width of the drop is unknown, its value being determined upon performing a detailed analysis of the shape of the drop. However, if one is concerned with the case when $(\Theta_A - \Theta_R)/\Theta_A \ll 1$ then it is fairly straightforward to show that

$$\omega \sim \left[\frac{12 V \sin^3\Theta_a}{\pi(1-\cos\Theta_A - \frac{1}{2}\cos\Theta_A \sin^2\Theta_A)}\right]^{\frac{1}{3}},$$

to lowest order for small drops. Hence, the critical angle of inclination, γ_c, can be determined without performing any detailed analysis of the shape of the drop. This is not possible for the current problem. Even in the case of small hysteresis the macroscopic force balance merely yields an identity. Hence, a detailed analysis cannot be avoided.

In §2.1 the variables are scaled and the appropriate boundary-value problem governing the motion of the fluids within and surrounding the drop are identified. The dependent variables are expanded asymptotically in terms of Θ_A as $\Theta_A \to 0$, and the equations governing the lower-order modes are determined in §2.2. In §2.3 the boundary-value problem governing the shape of the drop is derived. The problem is further simplified by expanding the shape of the drop asymptotically in terms of $(\Theta_A - \Theta_R)/\Theta_A$ as $(\Theta_A - \Theta_R)/\Theta_A \to 0$. The equations governing the lower-order modes appear in §2.4. In §3 the solution of the lowest-order mode is obtained. Finally, in §4 the lowest-order non-trivial value of dU/dZ as $\Theta_A \to 0$ and $(\Theta_A - \Theta_R)/\Theta_A \to 0$ is determined. Conclusions appear in §5.

2. Formulation

2.1. *Scaling and identification of boundary-value problem*

It will prove convenient when formulating the problem to make use of the rectangular Cartesian coordinate system (x, y, z) illustrated in figure 1. Here the z-axis points in the direction perpendicular to the surface of the solid and into the drop, the x-axis lies tangent to the solid surface coinciding with the direction of motion of the surrounding fluid far from the drop, and the origin is located on the solid surface equidistant from all four ends of the two straight-line segments on the contact line. A polar coordinate system (r, ϕ) defined on the $(z = 0)$-plane is also useful, refer again to figure 1. The orientation of the solid surface relative to gravity need not be specified since the analysis is restricted to small drops. That is to say, only the lowest-order mode represented by the limit as the Bond number $(\rho_d - \rho_s) g V^{\frac{2}{3}}/\sigma$ approaches zero is investigated.

It is natural to scale each immiscible fluid in the system differently. The fluid surrounding the drop will be considered first. Various choices exist for the lengthscale. Since the disturbance created by the presence of the drop on the constant rate of strain, $\mathrm{d}U/\mathrm{d}Z$, is of prime concern, the variables x, y, and z are scaled by a, the radius of the circle formed by the contact line under static conditions when the contact angle takes on a value Θ_A along its entire length. The velocity vector $\boldsymbol{u}$ is scaled by $(\mathrm{d}U/\mathrm{d}Z)\,a$, denoted by U_s. The pressure is scaled by $\mu_s\,U_s/a$. Upon introducing these scales into the Navier–Stokes and continuity equations one obtains

$$R_{es}\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{\nabla}}\hat{\boldsymbol{u}} = -\hat{\boldsymbol{\nabla}}\hat{P}+\hat{\nabla}^2\hat{\boldsymbol{u}}, \tag{2.1}$$

$$\hat{\boldsymbol{\nabla}}\cdot\hat{\boldsymbol{m}} = 0, \tag{2.2}$$

where R_{es} denotes the Reynolds number $\rho_s\,U_s\,a/\mu_s$,

$$\boldsymbol{\nabla}(\cdot) \equiv \boldsymbol{i}\frac{\partial(\cdot)}{\partial x}+\boldsymbol{j}\frac{\partial(\cdot)}{\partial y}+\boldsymbol{k}\frac{\partial(\cdot)}{\partial z},$$

μ_s denotes the dynamic viscosity of the fluid surrounding the drop, and the circumflex above a variable indicates that it has been made dimensionless with the scales indicated above. Finally, the boundary conditions away from the solid, and on that portion of the solid which bounds the surrounding fluid are, respectively,

$$\frac{\partial\hat{\boldsymbol{u}}}{\partial\hat{z}} = \boldsymbol{i} \quad \text{as } \hat{z}\to\infty, \tag{2.3}$$

$$\hat{\boldsymbol{u}} = \boldsymbol{0} \quad \text{at } \hat{z} = 0 \quad (\hat{x},\hat{y})\notin\hat{\mathscr{A}},\dagger \tag{2.4}$$

where $\mathscr{A}$ denotes the region on the solid surface in contact with the drop.

The scales of the variables describing the state of the fluid within the drop differ somewhat from those stated above. This is mainly a consequence of the confined geometry and the different physical characteristics of the fluid. The variables x and y are scaled by a; while, z is scaled with $a\Theta_A$ in anticipation of restricting the investigation to drops with small slopes, which in addition, greatly influences the scales of the remaining variables. The $\boldsymbol{i}$- and $\boldsymbol{j}$-component of the velocity, i.e. u and v respectively, are scaled by U_d; however, the $\boldsymbol{k}$-component, w, is scaled by $U_d\,\Theta_A$, where U_d denotes $U_s\mu_s\,\Theta_A/\mu_d$. The pressure is scaled by $\mu_a\,U_d/a\,\Theta_A^2$. Upon introducing these scales into the Navier–Stokes and continuity equations one obtains

$$\left.\begin{aligned} R_{ed}\,\Theta_A^2\left\{\bar{\boldsymbol{u}}_H\cdot\bar{\boldsymbol{\nabla}}_H\,\bar{\boldsymbol{u}}_H+\bar{w}\frac{\partial\bar{\boldsymbol{u}}_H}{\partial\bar{z}}\right\} &= -\bar{\boldsymbol{\nabla}}_H\,\bar{P}+\frac{\partial^2\bar{\boldsymbol{u}}_H}{\partial\bar{z}^2}+\Theta_A^2\,\bar{\boldsymbol{\nabla}}_H{}^2\bar{\boldsymbol{u}}_H,\\ R_{ed}\,\Theta_A^2\left\{\bar{\boldsymbol{u}}_H\cdot\bar{\boldsymbol{\nabla}}_H\,\bar{w}+\bar{w}\frac{\partial\bar{w}}{\partial\bar{z}}\right\} &= -\frac{1}{\Theta_A^2}\frac{\partial\bar{p}}{\partial\bar{z}}+\frac{\partial^2\bar{w}}{\partial\bar{z}^2}+\Theta_A^2\,\bar{\boldsymbol{\nabla}}_H{}^2\,\bar{w}, \end{aligned}\right\} \tag{2.5}$$

$$\bar{\boldsymbol{\nabla}}_H\cdot\bar{\boldsymbol{u}}_H+\frac{\partial\bar{w}}{\partial\bar{z}} = 0, \tag{2.6}$$

where R_{ed} denotes the Reynolds number $(p_d\,U_d a)/(\mu_d)$,

$$\boldsymbol{\nabla}_H(\cdot) \equiv \boldsymbol{i}\frac{\partial(\cdot)}{\partial x}+\boldsymbol{j}\frac{\partial(\cdot)}{\partial y},$$

$\boldsymbol{u}_H \equiv u\boldsymbol{i}+v\boldsymbol{j}$, μ_d denotes the dynamic viscosity of the fluid inside the drop, and the bar above a variable indicates that it has been made dimensionless with the scales indicated above. The remaining boundary conditions at the surface of the solid are given by

$$\bar{w} = \bar{\boldsymbol{u}}_H = 0 \quad \text{at} \quad \bar{z} = 0,\ (\bar{x},\bar{y})\in\bar{\mathscr{A}}. \tag{2.7a,b}$$

† The symbol $\notin$ denotes 'is not an element'.

The boundary conditions at the fluid interface of the drop and at the contact line are yet to be specified. The kinematic boundary condition is given by

$$\bar{\boldsymbol{u}}_{\mathrm{H}}\cdot\bar{\boldsymbol{\nabla}}_{\mathrm{H}}\bar{h}-\bar{w}=0 \quad \text{at} \quad \bar{z}=\bar{h}, \tag{2.8}$$

where $\bar{z}=\bar{h}(\bar{x},\bar{z})$ denotes the location of the fluid interface. Continuity of the velocity field requires

$$\hat{\boldsymbol{u}}_{\mathrm{H}}=\bar{\boldsymbol{u}}_{\mathrm{H}}\frac{\mu_{\mathrm{s}}}{\mu_{\mathrm{d}}}\Theta_{\mathrm{A}} \quad \text{and} \quad \hat{w}=\bar{w}\frac{\mu_{\mathrm{s}}}{\mu_{\mathrm{d}}}\Theta_{\mathrm{A}}^{2} \quad \text{at} \quad \bar{z}=\bar{h}. \tag{2.9a, b}$$

The dynamic boundary condition is given by

$$\bar{\boldsymbol{T}}(\boldsymbol{n})-\Theta_{\mathrm{A}}\hat{\boldsymbol{T}}(\boldsymbol{n})=\frac{\boldsymbol{n}}{Ca\,\bar{R}_{\mathrm{M}}} \quad \text{at}\, \bar{z}=\bar{h}, \tag{2.10}$$

where $\bar{\boldsymbol{T}}$ and $\hat{\boldsymbol{T}}$ denote the stress tensor made dimensionless with $\mu_{\mathrm{d}}\,U_{\mathrm{d}}/a\Theta_{\mathrm{A}}^{2}$ and $\mu_{\mathrm{s}}\,U_{\mathrm{s}}/a$ respectively; Ca denotes the usual capillary number scaled with Θ_{A}^{3}, $\mu_{\mathrm{d}}\,U_{\mathrm{s}}\mathrm{d}/\sigma\Theta_{\mathrm{A}}^{3}$; the mean radius of curvature is given by

$$\frac{1}{\bar{R}_{\mathrm{M}}}=\frac{\dfrac{\partial^{2}\bar{h}}{\partial\bar{x}^{2}}\left[1+\Theta_{\mathrm{A}}^{2}\left(\dfrac{\partial\bar{h}}{\partial\bar{y}}\right)^{2}\right]+\dfrac{\partial^{2}\bar{h}}{\partial\bar{y}^{2}}\left[1+\Theta_{\mathrm{A}}^{2}\left(\dfrac{\partial\bar{h}}{\partial\bar{x}}\right)^{2}\right]-2\Theta_{\mathrm{A}}^{2}\dfrac{\partial^{2}\bar{h}}{\partial\bar{x}\,\partial\bar{y}}\dfrac{\partial\bar{h}}{\partial\bar{x}}\dfrac{\partial\bar{h}}{\partial\bar{y}}}{\left[1+\Theta_{\mathrm{A}}^{2}\left(\dfrac{\partial\bar{h}}{\partial\bar{x}}\right)^{2}+\Theta_{\mathrm{A}}^{2}\left(\dfrac{\partial\bar{h}}{\partial\bar{y}}\right)^{2}\right]^{\frac{3}{2}}}, \tag{2.11}$$

and the unit outward normal is given by

$$\boldsymbol{n}=\frac{\boldsymbol{k}-\Theta_{\mathrm{A}}\dfrac{\partial\bar{h}}{\partial\bar{x}}\boldsymbol{i}-\Theta_{\mathrm{A}}\dfrac{\partial\bar{h}}{\partial\bar{y}}\boldsymbol{j}}{\left[1+\Theta_{\mathrm{A}}^{2}\left(\dfrac{\partial\bar{h}}{\partial\bar{x}}\right)^{2}+\Theta_{\mathrm{A}}^{2}\left(\dfrac{\partial\bar{h}}{\partial\bar{y}}\right)^{2}\right]^{\frac{1}{2}}}. \tag{2.12}$$

Finally, the boundary condition at the contact line is given by

$$\left.\begin{array}{l}\Theta=\begin{cases}\Theta_{\mathrm{A}} & -\phi_{\mathrm{A}}<\phi<\phi_{\mathrm{A}},\\ \Theta_{\mathrm{R}} & \phi_{\mathrm{R}}<\phi<2\pi-\phi_{\mathrm{R}},\end{cases}\\ \text{Contact line is a straight-line segment}\begin{cases}\phi_{\mathrm{A}}<\phi<\phi_{\mathrm{R}},\\ 2\pi-\phi_{\mathrm{R}}<\phi<2\pi-\phi_{\mathrm{A}},\end{cases}\end{array}\right\} \tag{2.13}$$

where the local value of the contact angle is given by

$$\sin\Theta=\boldsymbol{n}\cdot\boldsymbol{m} \quad \text{at}\,\{(\bar{x},\bar{y})|\bar{h}=0\}, \tag{2.14}$$

where the unit normal to the contact line pointing away from the drop and parallel to the surface of the solid is given by

$$\boldsymbol{m}=-\frac{\left[\dfrac{\partial\bar{h}}{\partial\bar{x}}\boldsymbol{i}+\dfrac{\partial\bar{h}}{\partial\bar{y}}\boldsymbol{j}\right]}{\left[\left(\dfrac{\partial\bar{h}}{\partial\bar{x}}\right)^{2}+\left(\dfrac{\partial\bar{h}}{\partial\bar{y}}\right)^{2}\right]^{\frac{1}{2}}} \quad \text{at}\,\{(\bar{x},\bar{y})|\bar{h}=0\}, \tag{2.15}$$

and the values of ϕ_{A} and ϕ_{R} are part of the solution.

The remaining requirement is the specification of the volume of the drop. It is given by

$$\frac{V}{a^{3}\Theta_{\mathrm{A}}}=\int_{\mathscr{A}}\bar{h}\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}. \tag{2.16}$$

It is important to realize that a solution does not exist for arbitrary values of the parameters appearing in the above governing equations and boundary conditions: Θ_A, Θ_R, $V/a^3\Theta_A$, Ca, ρ_d/ρ_s, μ_d/μ_s and $\rho_s a\sigma/\mu_s^2$, denoted by $\mathbb{D}$. [Note that $R_{es} = \Theta_A^2\, Ca\mathbb{D}$, and $R_{ed} = (\rho_d/\rho_s)(\mu_s/\mu_d)^2\,\Theta_A^3\, Ca\mathbb{D}$.] The local value of the contact angle will not satisfy the condition given in (2.13) unless the rate of strain in the surrounding fluid $\mathrm{d}U/\mathrm{d}Z$ takes on a specific value, the determination of which is the heart of the problem. That is to say, for given values of the parameters, the value of Ca, $\mu_s(\mathrm{d}U/\mathrm{d}Z)\,a/\sigma\Theta_A^2$, interpreted here as a dimensionless $\mathrm{d}U/\mathrm{d}Z$, must be determined for which a solution exists. Thus the principal objective of this study is the determination of the function $\mathscr{F}$, where

$$Ca = \mathscr{F}\left(\frac{\Theta_A-\Theta_R}{\Theta_A},\quad \Theta_A,\quad \frac{V}{a^3\Theta_A},\quad \frac{\rho_d}{\rho_s},\quad \frac{\mu_d}{\mu_s},\quad \mathbb{D}\right). \tag{2.17}$$

2.2. *Lubrication equations*

The analysis of the boundary-value problem presented in §2.1 simplifies considerably by restricting the value of Θ_A, and thus Θ_A, to be small. For this reason, only the lowest-order mode in the limit as Θ_A approaches zero will be investigated.

From a formal point of view, the only variables that will be determined which describe the motion of the surrounding fluid are $\hat{\boldsymbol{u}}_0$ and $\hat{p}_0$, where

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{u}}_0\left(\hat{\boldsymbol{x}};\ \frac{\Theta_A-\Theta_R}{\Theta_A},\ \frac{V}{a^3\Theta_A},\ \frac{\rho_d}{\rho_s},\ \frac{\mu_d}{\mu_s},\ \mathbb{D}\right)+O(\Theta_A) \tag{2.18a}$$

$$\hat{p} = \hat{p}_0\left(\hat{\boldsymbol{x}};\ \frac{\Theta_A-\Theta_R}{\Theta_A},\ \frac{V}{a^3\Theta_A},\ \frac{\rho_d}{\rho_s},\ \frac{\mu_d}{\mu_s},\ \mathbb{D}\right)+O(\Theta_A) \tag{2.18b}$$

valid in the limit as $\Theta_A \to 0$. Substituting the above into the Navier–Stokes and continuity equations (2.1) and (2.2) gives respectively

$$\mathbf{0} = -\hat{\boldsymbol{\nabla}}\hat{P}_0 + \hat{\boldsymbol{\nabla}}^2\hat{\boldsymbol{u}}_0, \tag{2.19}$$

$$\hat{\boldsymbol{\nabla}}\cdot\hat{\boldsymbol{u}}_0 = 0. \tag{2.20}$$

To lowest order, the boundary conditions (2.3) and (2.4) are

$$\frac{\partial\hat{\boldsymbol{u}}_0}{\partial\hat{z}} = \boldsymbol{i}\quad \text{as}\,\hat{z}\to\infty, \tag{2.21}$$

$$\hat{\boldsymbol{u}}_0 = \mathbf{0}\quad \text{at}\ \hat{z}=0,\quad (\hat{x},\hat{y})\notin\hat{\mathscr{A}}_0, \tag{2.22}$$

where $\hat{\mathscr{A}}_0$ denotes the limit of $\hat{\mathscr{A}}$ as $\Theta_A\to 0$.

The dependent variables describing the dynamics of the fluid within the drop are expanded asymptotically in a similar fashion

$$\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}_0\left(\bar{\boldsymbol{x}};\ \frac{\Theta_A-\Theta_R}{\Theta_A},\ \frac{V}{a^3\Theta_A},\ \frac{\rho_d}{\rho_s},\ \frac{\mu_d}{\mu_s},\ \mathbb{D}\right)+O(\Theta_A) \tag{2.23a}$$

$$\bar{p} = \bar{p}_0\left(\bar{\boldsymbol{x}};\ \frac{\Theta_A-\Theta_R}{\Theta_A},\ \frac{V}{a^3\Theta_A},\ \frac{\rho_d}{\rho_s},\ \frac{\mu_d}{\mu_s},\ \mathbb{D}\right)+O(\Theta_A) \tag{2.23b}$$

$$\bar{h} = \bar{h}_0\left(\bar{\boldsymbol{x}};\ \frac{\Theta_A-\Theta_R}{\Theta_A},\ \frac{V}{a^3\Theta_A},\ \frac{\rho_d}{\rho_s},\ \frac{\mu_d}{\mu_s},\ \mathbb{D}\right)+O(\Theta_A) \tag{2.23c}$$

Other solutions of (6.1 *a*) are given by

$$0 = A^2 + c_2 B^2 - c_3[R_1 - \mu]. \tag{6.3}$$

Substitution of A^2 from (6.3) into (6.1 *b*) shows that B must now satisfy the cubic equation

$$0 = B^3[1 - c_2 d_2] - B[d_3 R_1 - d_2 c_3 R_1 - d_2 c_3 \mu] - d_4. \tag{6.4}$$

We note that if $A^2 \geqslant 0$ then (6.3) gives

$$c_3[R_1 - \mu] - c_2 B^2 \geqslant 0.$$

The solutions of (6.4) satisfying the above inequality are shown as curves III and IV in figure 2 (*a*). Branch III bifurcates from curve I and then asymptotes to the solution of the perfect problem as shown. The corresponding behaviour of A is shown in figure 2 (*b*). We note that at the bifurcation point A is zero and dA/dR_1 is finite. Furthermore branch III is an unstable solution. The remaining solutions of (6.4) and the corresponding values of A are given by curve IV in figures 2 (*a*) and (*b*). These solutions begin at a point where dA/dR_1 and dB/dR_1 are both infinite. One branch of the curves is stable and the other unstable in both cases. However, we see that if R_1 is increased from zero then the motion follows curve I unless the motion is perturbed by a disturbance sufficiently large to enable A and B to tend to the stable singular points associated with the stable parts of IV.

Case (ii): $c_4 \neq 0$, $d_4 = 0$

The steady solutions of (4.22) now satisfy the equations

$$0 = A^3 + c_2 A B^2 - c_3(R_1 - \mu) A - c_4, \tag{6.5a}$$

$$0 = B(B^2 + d_2 A^2 - d_3 R_1). \tag{6.5b}$$

We assume, without any loss of generality, that the constant c_4 is positive. We can see from (6.5) that a possible solution is $B = 0$ and that A is then determined by the equation

$$0 = A^3 - c_3(R_1 - \mu) - c_4. \tag{6.6}$$

Alternatively we can see that (6.5 *b*) is satisfied if

$$B^2 = -d_2 A^2 + d_3 R_1 \tag{6.7}$$

and then A is determined by

$$0 = [1 - c_2 d_2] A^3 + A[-c_3 R_1 + c_3 \mu + c_2 d_3 R_1] - c_4. \tag{6.8}$$

If $B^2 \geqslant 0$ then solutions of (6.8) must satisfy the condition

$$d_2 A^2 < d_3 R_1.$$

The solutions of (6.6), (6.7) and (6.8) are shown in figures 3, 4 and 5. The two branches of the solution of (6.6) are shown in figures 3 (*b*), 4 (*b*) and 5 (*b*) and are in each case labelled I and II. One solution of (6.8) is the curve III shown in each of these figures. This curve bifurcates from curve II, so that B is virtually zero on III. In addition to III, (6.8) has the solution represented by IV in figures 3 (*b*), 4 (*b*) and 5 (*b*). We must distinguish between the following three cases:

(*a*) Curves I and IV do not intersect. This occurs when c_4 is sufficiently large and the amplitudes A and B for this case are shown in figure 3. In this case curve I is always

The expansion of the boundary condition at the contact line (2.13) is also rather straightforward. Substituting (2.23c) into (2.12), (2.13), (2.14), and (2.15) gives

$$\left.\begin{aligned} -|\bar{\nabla}_{\mathrm{H}}\bar{h}_0| &= \begin{cases} 1 & -\phi_{\mathrm{A}_0} < \phi < \phi_{\mathrm{A}_0}, \\ 1-\dfrac{\Theta_{\mathrm{A}}-\Theta_{\mathrm{R}}}{\Theta_{\mathrm{A}}} & \phi_{\mathrm{R}_0} < \phi < 2\pi-\phi_{\mathrm{R}_0}, \end{cases} \\ \text{Contact line is a straight line segment } &\begin{cases} \phi_{\mathrm{A}_0} < \phi < \phi_{\mathrm{R}_0}, \\ 2\pi-\phi_{\mathrm{R}_0} < \gamma < 2\pi-\phi_{\mathrm{A}_0}, \end{cases} \end{aligned}\right\} \quad (2.35)$$

evaluated at the position of the contact line $\bar{h}_0 = 0$. An explicit description of the location of the contact line in polar coordinates will also be used

$$\bar{r} = \bar{R}_0\left(\phi;\frac{\Theta_{\mathrm{A}}-\Theta_{\mathrm{R}}}{\Theta_{\mathrm{A}}},\frac{V}{a^3\Theta_{\mathrm{A}}}\right)+O(\Theta_{\mathrm{A}}). \quad (2.36)$$

The function $\bar{R}_0$ is obtained by solving the following equation

$$\bar{h}_0(\bar{R}_0(\phi),\phi) = 0. \quad (2.37)$$

Finally, the volume constraint (2.16) is given by

$$\frac{V}{a^3\Theta_{\mathrm{A}}} = \int_0^{2\pi}\int_0^{\bar{R}_0(\phi)} \bar{h}_0\,\bar{r}\,\mathrm{d}\bar{r}\,\mathrm{d}\phi. \quad (2.38)$$

2.3. *Solution of (0)-mode*

The motion of the fluids inside and surrounding the drop have been decoupled somewhat, at least to lowest order. That is to say, the motion of the fluid surrounding the drop can be determined separately, a direct consequence of the appearance of (2.30a,b). Once the motion of the surrounding fluid has been determined, it can then be used to solve for the motion of the fluid in the drop. Specifically, it enters the boundary-value problem through (2.32) and (2.33) as a known shear stress driving the motion of the fluid within the drop.

The solution of the boundary-value problem for the fluid surrounding the drop, (2.19), (2.20), (2.21), (2.22), and (2.30a,b), is simply

$$\hat{\boldsymbol{u}}_0 = \hat{z}\boldsymbol{i}, \quad \hat{p}_0 = \text{constant}. \quad (2.39a,b)$$

The boundary-value problem for the fluid within the drop is given by (2.24a,b), (2.25), (2.26a,b), (2.29), (2.31), (2.35), (2.37), (2.38), and

$$\left.\frac{\partial\bar{u}_0}{\partial\bar{z}}\right|_{\bar{z}=\bar{h}_0} = 1, \quad \left.\frac{\partial\bar{v}_0}{\partial\bar{z}}\right|_{\bar{z}=\bar{h}_0} = 0, \quad (2.40a,b)$$

where (2.40a,b) is a consequence of substituting (2.39a,b) into (2.32) and (2.33), respectively. Solving (2.24a,b), (2.25), (2.26a,b), (2.29), (2.31), and (2.40a,b) in the usual way gives

$$\bar{\nabla}_{\mathrm{H}}\cdot(\tfrac{1}{3}\bar{h}_0^3\bar{\nabla}_{\mathrm{H}}(\bar{\nabla}_{\mathrm{H}}^2\bar{h}_0)+\tfrac{1}{2}\boldsymbol{i}\bar{h}_0^2\mathscr{F}_0) = 0 \quad \text{for } (\bar{x},\bar{y})\in\bar{\mathscr{A}}_0. \quad (2.41)$$

It is useful to note that $\bar{\boldsymbol{u}}_{\mathrm{H}_0}$, and its height averaged local value, denoted by $\bar{\boldsymbol{Q}}_{\mathrm{H}_0}$, are given respectively by

$$\bar{\boldsymbol{u}}_{\mathrm{H}_0} = -\frac{1}{\mathscr{F}_0}\{\tfrac{1}{2}\bar{z}^2-\bar{z}\bar{h}_0\}\bar{\nabla}_{\mathrm{H}}(\bar{\nabla}_{\mathrm{H}}^2\bar{h}_0)+\boldsymbol{i}\bar{z}, \quad (2.42)$$

$$\bar{\mathbf{Q}}_{\mathrm{H}_0} = \frac{\bar{h}_0^2}{3\mathscr{F}_0}\bar{\nabla}_{\mathrm{H}}(\bar{\nabla}_{\mathrm{H}}^2\bar{h}_0)+\boldsymbol{i}\tfrac{1}{2}\bar{h}_0, \quad (2.43)$$

where
$$\overline{\mathbf{Q}}_{\mathrm{H}_0} \equiv \frac{1}{\bar{h}_0}\int_0^{\bar{h}_0} \bar{\boldsymbol{u}}_{\mathrm{H}_0}\,\mathrm{d}\bar{z}.$$

2.4. *Expansion of* $(\Theta_{\mathrm{A}}-\Theta_{\mathrm{R}})/\Theta_{\mathrm{A}}$

Obtaining the solution to (2.41), i.e. determining $\bar{h}_0$ and Ca for given values of the parameters $(\Theta_{\mathrm{A}}-\Theta_{\mathrm{R}})/\Theta_{\mathrm{A}}$ and $V/a^3\Theta_{\mathrm{A}}$ introduced by the boundary condition (2.35) and volume constraint (2.38), is a difficult non-linear boundary-value problem. The nonlinearity enters through both the governing partial differential equation and the boundary condition at the contact line, whose location, (2.37), itself is part of the solution. In order to make the problem tractable, an expansion is performed in the parameter $(\Theta_{\mathrm{A}}-\Theta_{\mathrm{R}})/\Theta_{\mathrm{A}}$, denoted hereafter as ϵ, in the limit as it approaches zero.

The prime objective is to determine (2.34), an expansion of which in ϵ gives
$$Ca \sim \mathscr{F}_{00}\left(\frac{V}{a^3\Theta_{\mathrm{A}}}\right)+\mathscr{F}_{01}\left(\frac{V}{a^3\Theta_{\mathrm{A}}}\right)\epsilon+\ldots, \tag{2.44}$$
valid in the limit as $\epsilon\rightarrow 0$. The expansion of the dependent variable $\bar{h}_0$ is not as straightforward. In order to determine an expression uniformly valid over the entire drop, two separate expansions must be performed (Dussan V. & Chow 1983), one valid over the regions in the immediate vicinity of the two straight-line segments on the sides of the drop, the other valid over the remainder of the drop. Only the latter expansion is pursued since it proves sufficient for evaluating $\mathscr{F}_{00}$ and $\mathscr{F}_{01}$. Hence, $\bar{h}_0$ is assumed to have the form
$$\bar{h}_0 \sim \bar{h}_{00}\left(\bar{r},\phi;\frac{V}{a^3\Theta_{\mathrm{A}}}\right)+\bar{h}_{0\mathrm{L}}\left(\bar{r},\phi;\frac{V}{a^3\Theta_{\mathrm{A}}}\right)\epsilon\ln\epsilon+\bar{h}_{01}\left(\bar{r},\phi;\frac{V}{a^3\Theta_{\mathrm{A}}}\right)\epsilon+\ldots, \tag{2.45}$$
valid in the limit as $\epsilon\rightarrow 0$. In fact, neither shall a solution be obtained for $\bar{h}_{0\mathrm{L}}$, nor a complete solution for $\bar{h}_{01}$ because they are unnecessary for determining the values of $\mathscr{F}_{00}$ and $\mathscr{F}_{01}$.

The problem simplifies quite a bit from the outset by simply acknowledging the fact that $\mathscr{F}_{00}\equiv 0$, i.e. when the static contact angle is unique, $\epsilon=0$, any motion in the surrounding fluid would cause the drop to travel across the surface of the solid. Substituting (2.44) and (2.45) into (2.42) and (2.43) gives, respectively,
$$\bar{\boldsymbol{u}}_{\mathrm{H}_0} \sim \boldsymbol{i}\bar{z}-\frac{1}{\mathscr{F}_{01}}\left\{\frac{\bar{z}^2}{2}-\bar{z}\bar{h}_{00}\right\}\overline{\boldsymbol{\nabla}}_{\mathrm{H}}\overline{\nabla}^2_{\mathrm{H}}\left[\frac{\bar{h}_{00}}{\epsilon}+\bar{h}_{01}\ln\epsilon+\bar{h}_{01}\right]+\ldots, \tag{2.46}$$
$$\epsilon\mathscr{F}_{01}\overline{\boldsymbol{Q}}_{\mathrm{H}_0} \sim \tfrac{1}{3}\bar{h}^2_{00}\overline{\boldsymbol{\nabla}}_{\mathrm{H}}(\overline{\nabla}^2_{\mathrm{H}}\bar{h}_{00})+\tfrac{1}{3}\bar{h}^2_{00}\overline{\boldsymbol{\nabla}}_{\mathrm{H}}(\overline{\nabla}^2_{\mathrm{H}}\bar{h}_{0\mathrm{L}})\epsilon\ln\epsilon+[\![\tfrac{1}{3}\bar{h}^2_{00}\overline{\boldsymbol{\nabla}}_{\mathrm{H}}(\overline{\nabla}^2_{\mathrm{H}}\bar{h}_{01})$$
$$+\tfrac{2}{3}\bar{h}_{00}\bar{h}_{01}\overline{\boldsymbol{\nabla}}_{\mathrm{H}}(\overline{\nabla}^2_{\mathrm{H}}\bar{h}_{00})+\boldsymbol{i}\tfrac{1}{2}\bar{h}_{00}\mathscr{F}_{01}]\!]\epsilon+\ldots, \tag{2.47}$$
valid in the limit as $\epsilon\rightarrow 0$. Since $\overline{\boldsymbol{Q}}_{\mathrm{H}0}=O(1)$ as $\epsilon\rightarrow 0$, it follows that to order 1 and $\epsilon\ln\epsilon$, respectively
$$\overline{\nabla}^2_{\mathrm{H}}\bar{h}_{00}=A_{00},\quad \overline{\nabla}^2\bar{h}_{0\mathrm{L}}=A_{0\mathrm{L}} \tag{2.48a,b}$$
where A_{00}, and $A_{0\mathrm{L}}$ are constants to be determined. Substituting (2.44) and (2.45) into (2.41) gives
$$\overline{\boldsymbol{\nabla}}_{\mathrm{H}}\cdot\{\tfrac{1}{3}\bar{h}^3_{00}\overline{\boldsymbol{\nabla}}_{\mathrm{H}}(\overline{\nabla}^2_{\mathrm{H}}\bar{h}_{01})+\boldsymbol{i}\tfrac{1}{2}\bar{h}_{00}\mathscr{F}_{01}\}=0, \tag{2.49}$$
to order ϵ. It is helpful when expanding both (2.35) and (2.38) to use the explicit expression for the location of the contact line (2.36). The expansion given by (2.45) implies
$$R_0 \sim R_{00}\left(\phi;\frac{V}{a^3\Theta_{\mathrm{A}}}\right)+R_{0\mathrm{L}}\left(\phi;\frac{V}{a^3\Theta_{\mathrm{A}}}\right)\epsilon\ln\epsilon+R_{01}\left(\phi;\frac{V}{a^3\Theta_{\mathrm{A}}}\right)\epsilon+\ldots, \tag{2.50}$$

valid in the limit as $\epsilon \to 0$. The development can be further simplified by anticipating the fact that to lowest order the shape of the drop, $\bar{h}_{00}$, and hence, the shape of the contact line, R_{00}, is independent of ϕ. Combining this with the definition of a appearing at the beginning of §2.1 implies

$$R_{00} = 1. \tag{2.51}$$

Substituting (2.45), (2.50) and (2.51) into (2.37), and making the dependence of the resulting expression on ϵ explicit by performing an expansion in $\bar{r}$ about 1 similar in spirit to (2.28), gives to lowest orders

$$\bar{h}_{00}(1,\phi) = 0, \quad \bar{h}_{0\mathrm{L}}(1,\phi) + R_{0\mathrm{L}} \frac{\partial \bar{h}_{00}}{\partial \bar{r}}\bigg|_{(1,\phi)} = 0, \quad \bar{h}_{01}(1,\phi) + R_{01} \frac{\partial \bar{h}_{00}}{\partial \bar{r}}\bigg|_{(1,\phi)} = 0. \tag{2.52a, b, c}$$

In a similar way, substituting (2.45), (2.50) and (2.51) into an expansion of (2.35) about $\bar{r} = 1$ gives to lowest orders

$$\frac{\partial \bar{h}_{00}}{\partial \bar{r}} = -1 \quad \text{at}\, \bar{r} = 1 \quad (0 \leqslant \phi \leqslant 2\pi), \tag{2.53}$$

$$R_{01} \frac{\partial^2 \bar{h}_{00}}{\partial \bar{r}^2}\bigg|_{\bar{r}=1} + \frac{\partial \bar{h}_{01}}{\partial \bar{r}}\bigg|_{\bar{r}=1} = \begin{cases} 0 & -\frac{1}{2}\pi < \phi < \frac{1}{2}\pi, \\ 1 & \frac{1}{2}\pi < \phi < \frac{3}{2}\pi, \end{cases} \tag{2.54}$$

where it has been assumed that

$$\phi_{\mathrm{A}_0} \sim \tfrac{1}{2}\pi + \phi_{\mathrm{A}_{01}} \epsilon + \dots,$$
$$\phi_{\mathrm{R}_0} \sim \tfrac{1}{2}\pi + \phi_{\mathrm{R}_{01}} \epsilon + \dots.$$

Substituting (2.45), (2.50) and (2.51) into the volume constraint (2.38) gives to lowest orders

$$\frac{V}{a^3 \Theta_{\mathrm{A}}} = 2\pi \int_0^1 \bar{h}_{00} \bar{r}\, \mathrm{d}\bar{r}, \quad 0 = \int_0^1 \int_0^{2\pi} \bar{h}_{0\mathrm{L}} \bar{r}\, \mathrm{d}\phi\, \mathrm{d}\bar{r}, \quad 0 = \int_0^1 \int_0^{2\pi} \bar{h}_{01} \bar{r}\, \mathrm{d}\phi\, \mathrm{d}\bar{r}. \tag{2.55a, b, c}$$

3. Solution of (00)-mode

The boundary-value problem governing $\bar{h}_{00}$ is given by (2.48*a*), (2.52*a*), (2.53), and (2.55*a*), along with the restriction that $\bar{h}_{00}$ evaluated at the centre of the drop, $\bar{r} = 0$, is bounded. Obtaining the solution is straightforward. It is

$$\bar{h}_{00} = \tfrac{1}{2}(1 - \bar{r}^2), \quad A_{00} = -2. \tag{3.1a, b}$$

Substituting (31.1*a*) into the volume constraint (2.55*a*) gives an expression for the scale a

$$a^3 = \frac{4}{\pi} \frac{V}{\Theta_{\mathrm{A}}}. \tag{3.2}$$

4. Solution for $\mathscr{F}_{01}$

4.1. *Determination of the form of* $\bar{\nabla}^2_{\mathrm{H}} \bar{h}_{01}$

As stated in §2.4, the value of $\mathscr{F}_{01}$ can be determined without obtaining the complete solution for $\bar{h}_{01}$. The analysis divides into two parts. The first consists of identifying an explicit form for $\bar{\nabla}^2_{\mathrm{H}} \bar{h}_{01}$. The second consists of evaluating conditions necessary and sufficient for the existence of a solution for $\bar{h}_{01}$. It is these conditions which give rise to an expression for $\mathscr{F}_{01}$.

Although not essential, it is instructive to establish a precise relationship between $\bar{\nabla}_{\mathrm{H}}^{2}\bar{h}_{01}$ and the pressure field within the drop. A new dimensionless expression for the pressure $\mathscr{P}$ is introduced scaled with $\sigma\Theta_{\mathrm{A}}/a$. The scale represents a characteristic value of the jump in pressure across the fluid interface due to the presence of surface tension. The two dimensionless pressures are related by $\bar{P} = \mathscr{P}/Ca$. It can readily be shown that

$$-\mathscr{P}_{01} = \bar{\nabla}_{\mathrm{H}}^{2}\bar{h}_{01}, \tag{4.1}$$

where
$$\mathscr{P} \sim \mathscr{P}_{00}\left(\bar{r},\phi;\frac{V}{a^{3}\Theta_{\mathrm{A}}}\right)+\mathscr{P}_{0\mathrm{L}}\left(\bar{r},\phi;\frac{V}{a^{3}\Theta_{\mathrm{A}}}\right)\epsilon\ln\epsilon+\mathscr{P}_{01}\left(\bar{r},\phi;\frac{V}{a^{3}\Theta_{\mathrm{A}}}\right)\epsilon+\ldots,$$

valid in the limit as $\Theta_{\mathrm{A}}\rightarrow 0$ and $\epsilon\rightarrow 0$. Note that the value of $\mathscr{P}_{0}$ does not depend on $\bar{z}$, a direct consequence of (2.24*b*).

Using polar coordinates and the solution for $\bar{h}_{00}$ presented in (3.1*a*), (2.49) becomes

$$\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left\{(1-\bar{r}^{2})^{3}\,\bar{r}\frac{\partial\mathscr{P}_{01}}{\partial\bar{r}}\right\}+\frac{1}{\bar{r}^{2}}\frac{\partial}{\partial\phi}\left\{(1-\bar{r}^{2})^{3}\frac{\partial\mathscr{P}_{01}}{\partial\phi}\right\} = 12\mathscr{F}_{01}\,\bar{r}(\bar{r}^{2}-1)\cos\phi, \tag{4.2}$$

where $\boldsymbol{i} = \hat{\boldsymbol{r}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi$. A solution is sought of the form

$$\frac{\mathscr{P}_{01}}{\mathscr{F}_{01}} = \sum_{n=0}^{\infty} \mathbb{P}_{n}(\bar{r})\cos n\phi. \tag{4.3}$$

Substituting (4.3) into (4.2) gives

$$\frac{1}{\bar{r}}\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left\{(1-\bar{r}^{2})^{3}\,\bar{r}\frac{\mathrm{d}\mathbb{P}_{0}}{\mathrm{d}\bar{r}}\right\} = 0, \tag{4.4}$$

$$\frac{1}{\bar{r}}\frac{\mathrm{d}}{\mathrm{d}\bar{r}}\left\{(1-\bar{r}^{2})^{3}\,\bar{r}\frac{\mathrm{d}\mathbb{P}_{n}}{\mathrm{d}\bar{r}}\right\}-\frac{n^{2}(1-\bar{r}^{2})^{3}}{\bar{r}^{2}}\mathbb{P}_{n} = \begin{cases}12\bar{r}(\bar{r}^{2}-1) & \text{for } n=1,\\ 0 & \text{for } n=2,3,\ldots.\end{cases} \tag{4.5}$$

The boundary conditions appropriate for these equations are

$$\mathbb{P}_{0} \text{ is bounded at } \bar{r}=0,\quad \mathbb{P}_{n}=0 \text{ at } \bar{r}=0 \quad \text{for } n=1,2,3,\ldots, \tag{4.6a,b}$$

(4.6*b*) resulting from the requirement that the function $\mathscr{P}_{01}$ must be continuous at $\bar{r}=0$, and

$$(1-\bar{r}^{2})^{2}\frac{\mathrm{d}\mathbb{P}_{n}}{\mathrm{d}\bar{r}}\rightarrow 0 \quad \text{as } \bar{r}\rightarrow 1 \quad \text{for } n=0,1,2,\ldots, \tag{4.7}$$

a direct requirement that $\bar{\boldsymbol{u}}\rightarrow 0$ uniformly as $\boldsymbol{x}$ approaches the location of the static contact line.

It follows directly from (4.4), (4.6*a*) and (4.7) that

$$\mathbb{P}_{0} = \text{constant}. \tag{4.8}$$

The solutions for $\{\mathbb{P}_{n}: n=2,3\ldots\}$ are also obtained in a rather straightforward manner. Multiplying (4.5) by $\bar{r}\mathbb{P}_{n}$, integrating with respect to $\bar{r}$ over the interval [0, 1], and making repeated use of (4.7) and (4.6*b*) gives

$$n^{2} = -\frac{\displaystyle\int_{0}^{1}(1-\bar{r}^{2})^{3}\,\bar{r}\left(\frac{\mathrm{d}\mathbb{P}_{n}}{\mathrm{d}\bar{r}}\right)^{2}\mathrm{d}\bar{r}}{\displaystyle\int_{0}^{1}(1-\bar{r}^{2})^{3}\,\mathbb{P}_{n}^{2}\,\mathrm{d}\bar{r}}.$$

Since n^{2} is both real and positive, and $\mathbb{P}_{n}$ is also real, it must be concluded that

$$\{\mathbb{P}_{n}=0: n=2,3,\ldots\}. \tag{4.9}$$

The solution for $\mathbb{P}_{1}$ is given in §4.3.

4.2. *Determination of* $\mathscr{F}_{01}$

The boundary-value problem has been reduced to

$$-\bar{\nabla}_{\mathrm{H}}^2 \bar{h}_{01} = \mathscr{F}_{01}\{\mathbb{P}_0 + \mathbb{P}_1(\bar{r}) \cos\phi\}, \tag{4.10}$$

obtained by substituting (4.3), (4.8), and (4.9) into (4.1), subject to the boundary condition

$$\bar{h}_{01} - \frac{\partial \bar{h}_{01}}{\partial \bar{r}} = \begin{cases} 0 & \text{for } -\tfrac{1}{2}\pi < \phi < \tfrac{1}{2}\pi, \bar{r} = 1, \\ -1 & \text{for } \tfrac{1}{2}\pi < \phi < \tfrac{3}{2}\pi, \bar{r} = 1, \end{cases} \tag{4.11}$$

obtained by substituting (3.1*a*) into (2.52*c*) and (2.54). It is desirable to transform the problem defined by (4.10) and (4.11) into one involving a linear operator. This is accomplished by expressing $\bar{h}_{01}$ as

$$\bar{h}_{01} \equiv \mathscr{H} + \mathscr{H}_{\mathrm{p}},$$

where $\mathscr{H}_{\mathrm{p}}$ represents *any* function continuous on the open set $\{(\bar{r}, \phi)|0 < \phi < 2\pi, \bar{r} < 1\}$ and satisfying (4.11) with h_{01} replaced by $\mathscr{H}_{\mathrm{p}}$; and $\mathscr{H}$ satisfies

$$-\bar{\nabla}_{\mathrm{H}}^2 \mathscr{H} = \mathscr{F}_{01}\{\mathbb{P}_0 + \mathbb{P}_1(\bar{r}) \cos\phi\} + \bar{\nabla}_{\mathrm{H}}^2 \mathscr{H}_{\mathrm{p}}, \tag{4.12}$$

subject to the boundary condition

$$\mathscr{H} - \frac{\partial \mathscr{H}}{\partial \bar{r}} = 0 \quad \text{for } 0 \leqslant \phi \leqslant 2\pi, \bar{r} = 1. \tag{4.13}$$

Fredholm's Alternative may now be used to establish the conditions under which a solution exists to (4.12) and (4.13).

The necessary and sufficient conditions arising from Fredholm's Alternative establishing the existence of a solution to (4.12) and (4.13) requires that the function appearing on the right-hand side of (4.12) be orthogonal to every element of the null space of the adjoint linear operator. Introducing the inner product

$$\langle f, k \rangle \equiv \int_0^{2\pi} \int_0^1 f(\bar{r}, \phi)\, k(\bar{r}, \phi)\, \bar{r}\, \mathrm{d}\bar{r}\, \mathrm{d}\phi,$$

it is well known that the linear operator defined by (4.12) and (4.13) is self-adjoint. It is also straightforward to establish that the null space of the operator is spanned by the two linearly independent functions

$$\{\bar{r} \sin\phi, \bar{r} \cos\phi\}. \tag{4.14}$$

Hence, a solution exists if and only if

$$\langle \mathscr{F}_{01}\{\mathbb{P}_0 + \mathbb{P}_1(\bar{r}) \cos\phi\} + \bar{\nabla}_{\mathrm{H}}^2 \mathscr{H}_{\mathrm{p}}, \mathscr{G} \rangle = 0, \tag{4.15}$$

where $\mathscr{G}$ denotes any linear combination of the two functions appearing in (4.14). Using the identity

$$\langle \bar{\nabla}_{\mathrm{H}}^2 \mathscr{H}_{\mathrm{p}}, \mathscr{G} \rangle = \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \mathscr{G}\Big|_{\bar{r}=1} \mathrm{d}\phi,$$

it follows directly from (4.15) that

$$\mathscr{F}_{01} = \frac{2}{\pi \int_0^1 \mathbb{P}_1(\bar{r})\, \bar{r}^2\, \mathrm{d}\bar{r}}, \tag{4.16}$$

when $\mathscr{G} = \bar{r}\cos\phi$. The identity $0 = 0$ is obtained when $\mathscr{G} = \bar{r}\sin\phi$.

4.3. *Evaluation of* $\mathbb{P}_1$

All that remains is to evaluate

$$\int_0^1 \mathbb{P}_1(\bar{r})\,\bar{r}^2\,\mathrm{d}\bar{r}.$$

This requires obtaining a solution to (4.5), a singular linear second-order ordinary differential equation with variable coefficients, subject to the boundary conditions (4.6*b*) and(4.7). Two independent numerical techniques have been used to analyse this two-point boundary-value problem, the results of which are in very good mutual agreement.

It is useful to begin by analysing $\mathbb{P}_1$ in the neighbourhoods of $\bar{r} = 0$ and $\bar{r} = 1$ since (4.5) is singular at both of these end points. It is straightforward to show that near $\bar{r} = 0$, $\mathbb{P}_1$ is given by

$$\mathbb{P}_1 = -\{\tfrac{3}{2}\bar{r}^3 + \tfrac{17}{8}\bar{r}^5 + \tfrac{507}{192}\bar{r}^7 + \dots\} + C\{\bar{r} + \tfrac{3}{4}\bar{r}^3 + \tfrac{13}{16}\bar{r}^5 + \tfrac{351}{384}\bar{r}^7 + \dots\}, \tag{4.17}$$

where C denotes an unknown constant, and (4.17) satisfies the boundary condition (4.6*b*). The simplicity of (4.17) reflects the fact that the singular nature of (4.5) at $\bar{r} = 0$ arises from the degeneracy of the polar coordinate system and not from the physics of the problem. The situation is different near $\bar{r} = 1$. Here, a convenient form to assume for $\mathbb{P}_1$ is

$$\mathbb{P}_1 = \sum_{n=-2}^{\infty} (1-\bar{r})^n \{B_{n\mathrm{L}} \ln(1-\bar{r}) + B_n\}. \tag{4.18}$$

Substituting (4.18) into (4.5), and performing some straightforward algebraic manipulations gives

$$\begin{aligned}\mathbb{P}_1 = {} & -\tfrac{3}{2}\ln(1-\bar{r}) - \tfrac{5}{4}(1-\bar{r}) - (1-\bar{r})^2\{\tfrac{3}{16}\ln(1-\bar{r}) + \tfrac{31}{64}\} \\ & -(1-\bar{r})^3\{\tfrac{21}{80}\ln(1-\bar{r}) + 0.3694\} - \tfrac{117}{384}(1-\bar{r})^4\ln(1-\bar{r}) + \dots \\ & + B_0[\![1 + \tfrac{1}{8}(1-\bar{r})^2 + \tfrac{7}{40}(1-\bar{r})^3 + \dots]\!] + B_{-2}\bigg[\!\!\bigg[\frac{1}{(1-\bar{r})^2} \\ & + \frac{5}{1-\bar{r}} - \tfrac{15}{2}\ln(1-\bar{r}) - \tfrac{31}{4}(1-\bar{r}) - \tfrac{15}{16}(1-\bar{r})^2\ln(1-\bar{r}) + \dots\bigg]\!\!\bigg],\end{aligned} \tag{4.19}$$

where the expressions multiplying B_0 and B_{-2} represent the two linearly independent solutions to the homogeneous form of (4.5). The boundary condition (4.7) requires $B_2 = 0$. Hence, it is apparent that the solution near $\bar{r} = 1$ is singular even though the contact line is static, a fact which could have been anticipated at the outset due to the differing forms of the boundary conditions assumed at the free and solid surfaces of the drop.

The boundary-value problem can now be reformulated for the purpose of facilitating its numerical analysis by taking advantage of the above derived characteristics of $\mathbb{P}_1$ near $\bar{r} = 1$. A new dependent variable, $\mathbb{P}_\mathrm{D}$, is introduced defined by

$$\mathbb{P}_\mathrm{D} \equiv \mathbb{P}_1 + \tfrac{3}{2}\ln(1-\bar{r}). \tag{4.20}$$

Substituting (4.20) into (4.5) gives

$$\begin{aligned}&\frac{1}{\bar{r}}\frac{\mathrm{d}}{\mathrm{d}F}\left((1-\bar{r}^2)^3\,\bar{r}\,\frac{\mathrm{d}\mathbb{P}_\mathrm{D}}{\mathrm{d}\bar{r}}\right) - \frac{(1-\bar{r}^2)^3}{\bar{r}^2}\mathbb{P}_\mathrm{D} \\ &\quad = -\tfrac{3}{2}(1-\bar{r}^2)\bigg[\!\!\bigg[\frac{(1+3\bar{r})(1+\bar{r})(1-2\bar{r})}{\bar{r}} + \frac{(1-\bar{r}^2)^2}{\bar{r}^2}\ln(1-\bar{r})\bigg]\!\!\bigg] - 12\bar{r}(1-\bar{r}^2),\end{aligned} \tag{4.21}$$

$\bar{r}$	$\mathbb{P}_{\mathrm{D}}$	$\bar{r}$	$\mathbb{P}_{\mathrm{D}}$	$\bar{r}$	$\mathbb{P}_{\mathrm{D}}$
0	0	0.38	0.6210	0.74	1.107
0.02	0.0369	0.40	0.6499	0.76	1.132
0.04	0.0732	0.42	0.6786	0.78	1.157
0.06	0.1090	0.44	0.7071	0.80	1.182
0.08	0.1442	0.46	0.7352	0.82	1.207
0.10	0.1790	0.48	0.7631	0.84	1.232
0.12	0.2132	0.50	0.7907	0.86	1.256
0.14	0.2470	0.52	0.8182	0.88	1.281
0.16	0.2803	0.54	0.8453	0.90	1.305
0.18	0.3132	0.56	0.8723	0.92	1.330
0.20	0.3456	0.58	0.8991	0.94	1.355
0.22	0.3776	0.60	0.9256	0.96	1.379
0.24	0.4093	0.62	0.9520	0.98	1.404
0.26	0.4406	0.64	0.9781	1.00	1.429
0.28	0.4715	0.66	1.004		
0.30	0.5020	0.68	1.030		
0.32	0.5322	0.70	1.056		
0.34	0.5621	0.72	1.081		
0.36	0.5917				

TABLE 1

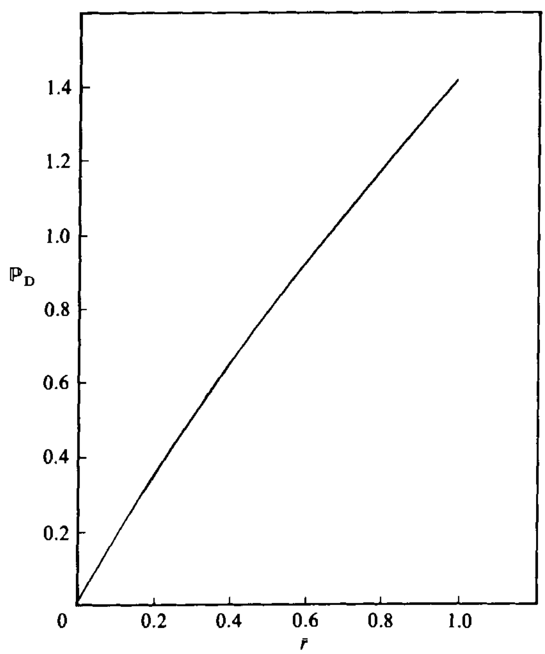


FIGURE 2. The dependence of $\mathbb{P}_{\mathrm{D}}$ on $\bar{r}$.

subject to the boundary conditions

$$\mathbb{P}_{\mathrm{D}} = 0 \quad \text{at}\, \bar{r} = 0, \tag{4.22}$$

$$\frac{\mathrm{d}\mathbb{P}_{\mathrm{D}}}{\mathrm{d}\bar{r}} = \tfrac{5}{4} \quad \text{at}\, \bar{r} = 1, \tag{4.23}$$

where (4.22) comes directly from (4.6*b*), and (4.23) can be deduced from (4.19).

	$\int_0^1 \mathbb{P}_1 \bar{r}^2 \, \mathrm{d}\bar{r}$	$\int_0^1 \left\{(1-\bar{r}^2)^2 - \frac{(1-\bar{r}^2)^3}{3\bar{r}} \mathbb{P}_1\right\} \mathrm{d}\bar{r}$
Sullivan	1.287	0.00023
Xu	1.290	0.00052

TABLE 2

Two numerical solutions have been obtained to (4.21), (4.22), and (4.23). Mr Christopher Sullivan performed an analysis using a finite element method outlined in Sullivan (1984). Mr Jian-Jun Xu used a computer program called SUPORT. Since both results agree to at least four significant figures uniformly over the entire interval $\bar{r} \in [0, 1]$, only one set is presented in table 1 and illustrated in figure 2. The values of

$$\int_0^1 \mathbb{P}_1(\bar{r}) \, \bar{r}^2 \, \mathrm{d}\bar{r},$$

and

$$\int_0^1 \left\{(1-\bar{r}^2)^2 - \frac{(1-\bar{r}^2)^3}{3\bar{r}} \mathbb{P}_1(\bar{r})\right\} \mathrm{d}\bar{r}$$

are given in table 2. The latter integral should be zero, a result which can easily be established by integrating (4.5) over the interval $\bar{r} \in [0, 1]$ and using (4.6*b*) and (4.7).

5. Conclusions

The heart of the problem has been to determine the maximum value of $\mathrm{d}U/\mathrm{d}Z$ consistnt with a stationary drop. Upon combining the definition of *Ca* with (2.42) and (4.16), one obtains to lowest order in $\Theta_{\mathrm{A}} \rightarrow 0$ and $(\Theta_{\mathrm{A}} - \Theta_{\mathrm{R}})/\Theta_{\mathrm{A}} \rightarrow 0$

$$\frac{\mathrm{d}U}{\mathrm{d}Z} \sim \frac{0.452\, \sigma \Theta_{\mathrm{A}}^{\frac{4}{3}} (\Theta_{\mathrm{A}} - \Theta_{\mathrm{R}})}{V^{\frac{1}{3}} \mu_{\mathrm{s}}},$$

where it has been assumed that

$$\int_0^1 \mathbb{P}_1 \bar{r}^2 \, \mathrm{d}\bar{r} \approx 1.3.$$

It is of interest to note that the critical value of $\mathrm{d}U/\mathrm{d}Z$ is independent of the viscosity of the drop μ_{d}. This can be viewed as following directly from the relationship defining the characteristic speed of the fluid in the drop U_{d}, i.e. $\mu_{\mathrm{d}} U_{\mathrm{d}} = \mu_{\mathrm{s}} U_{\mathrm{s}} \Theta_{\mathrm{A}}$, motivated by the shear stress boundary condition at the fluid–fluid interface. Consequently, the appearances of μ_{d} in both the definition of *Ca*, $\mu_{\mathrm{d}} U_{\mathrm{d}}/\sigma\Theta_{\mathrm{A}}^3$, and in the scale for the pressure within the drop, $\mu_{\mathrm{d}} U_{\mathrm{d}}/a\Theta_{\mathrm{A}}^2$, are 'disguised' in the form $U_{\mathrm{s}}\mu_{\mathrm{s}}/\sigma\Theta_{\mathrm{A}}^2$ and, $U_{\mathrm{s}}\mu_{\mathrm{a}}/a\Theta_{\mathrm{A}}$, respectively. While the value of μ_{d} does not affect the critical value of $\mathrm{d}U/\mathrm{d}Z$, it does determine the duration of time necessary for the drop to accommodate to any changes in the shear stress imposed by the motion of the surrounding fluid, the appropriate timescale being a/U_{d}. Thus, it would take a relatively long time for a very viscous drop to fully respond to the conditions dictated by the surrounding fluid. Everything else being equal, two drops composed of fluids with greatly different viscosities have identical crictical configurations.

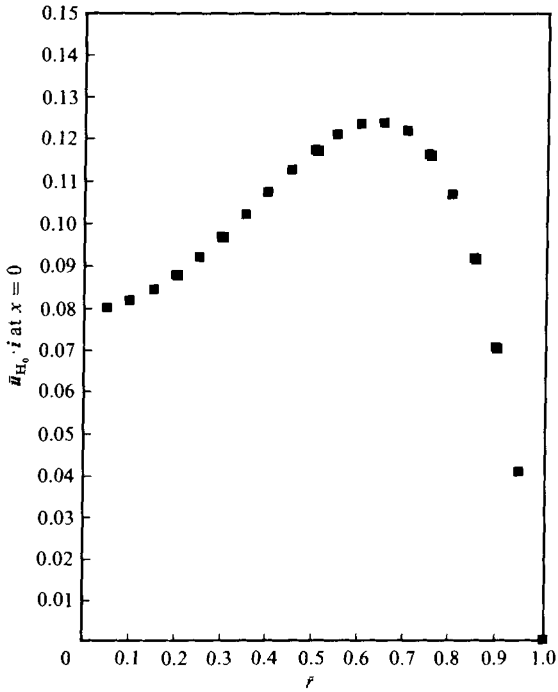


FIGURE 3. The variation of $\bar{u}_{H_0} \cdot \boldsymbol{i}$ at the fluid interface along $\bar{x} = 0$.

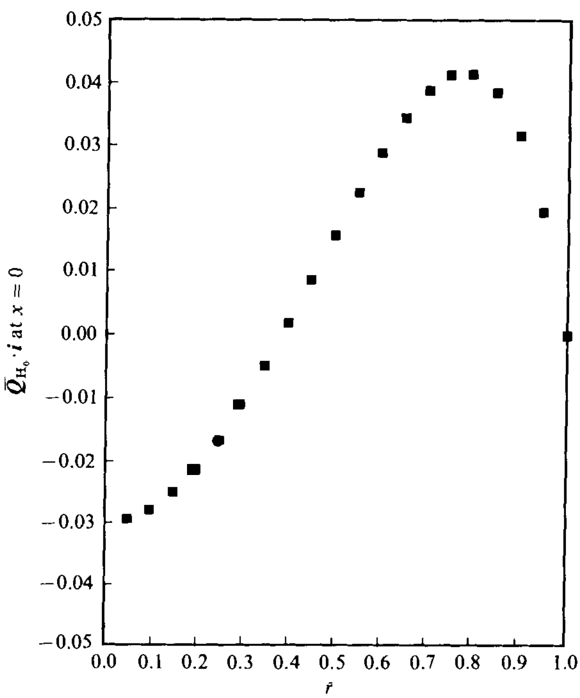


FIGURE 4. The variation of $\bar{Q}_{H_0} \cdot \boldsymbol{i}$ along $\bar{x} = 0$.

It is also of interest to examine the velocity field of the fluid within the drop. Substituting (2.48*a*, *b*) and (4.10) into (2.46) gives

$$\bar{\boldsymbol{u}}_{\mathrm{H}_0} \sim \boldsymbol{i}\bar{z} + \{\tfrac{1}{2}\bar{z}^2 - \bar{z}\bar{h}_{00}\}\,\bar{\boldsymbol{\nabla}}_{\mathrm{H}}(\mathbb{P}_1 \cos\phi).$$

As might have been anticipated, it can readily be shown that the $\boldsymbol{i}$-component of the velocity of the fluid is positive everywhere on the fluid interface of the drop, i.e. the fluid at the surface of the drop is moving in the same direction as that of its surroundings. An evaluation of $\bar{\boldsymbol{u}}_{\mathrm{H}_0}\cdot\boldsymbol{i}$ at $\bar{z} = \bar{h}_{00}$ and $\bar{x} = 0$ is given in figure 3. Although its value is everywhere positive, there appears to be an unanticipated minimum at $\bar{y} = 0$. However, this can easily be explained by a more detailed examination of the velocity field. The induced flow at the fluid interface of the drop must result in a net accumulation of fluid on the downstream end of the drop, as indicated in figure 1(*a*), causing the curvature of the fluid interface to have its greatest value there. The varying value of the curvature along the length of the drop creates the gradient in pressure necessary to cause a return flow in the $-\boldsymbol{i}$ direction so that the continuity equation is not violated, i.e.

$$\int_{\omega_{\bar{x}}} \bar{h}_{00}\,\bar{\boldsymbol{Q}}_{\mathrm{H}_0}\cdot\boldsymbol{i}\,\mathrm{d}\bar{y} = 0,$$

where $\omega_{\bar{x}}$ denotes the width of the drop at any fixed value of $\bar{x}$. Substituting (2.48*a*, *b*) and (4.10) into (2.47) gives

$$\bar{\boldsymbol{Q}}_{\mathrm{H}_0} = -\tfrac{1}{3}\bar{h}_{00}^2\,\bar{\boldsymbol{\nabla}}_{\mathrm{H}}(\mathbb{P}_1 \cos\phi) + \tfrac{1}{2}\bar{h}_{00}\,\boldsymbol{i}.$$

Upon evaluating $\bar{\boldsymbol{Q}}_{\mathrm{H}_0}\cdot\boldsymbol{i}$ along $\bar{x} = 0$, one finds that the return flow occurs primarily near the centre of the drop, see figure 4. This is not surprising because the centre of the drop represents the thickest region, hence offering the path of least resistance for the returning flow of fluid. The minimum in $\bar{\boldsymbol{u}}_{\mathrm{H}_0}\cdot\boldsymbol{i}$ along $\bar{x} = 0$ at $\bar{y} = 0$ merely reflects the fact that a substantial quantity of fluid is moving in the $-\boldsymbol{i}$ direction beneath the fluid interface.

The most restrictive assumptions in this analysis are the small slope approximation, $\Theta_{\mathrm{A}} \ll 1$, and, as in the two previous studies in this series, the necessity for small contact angle hysteresis, $(\Theta_{\mathrm{A}} - \Theta_{\mathrm{R}})/\Theta_{\mathrm{A}} \ll 1$.

It is a pleasure to acknowledge Mr Jian-Jun Xu at Northwestern University and Mr Christopher F. Sullivan at Lehigh University who are responsible for the results appearing in figure 2 and tables 1 and 2.

I am grateful for the support provided by the Guggenheim Foundation without which this work would not have been possible.

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